Permutations, variations and combinations

Combinatorics [1] deals with finite sets and its basic objects are permutations, variations and combinations (and they concern operations as well as the results of these operations). Since there is always possible to identify m- and n-elements set with the sets $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, it is enough to consider these sets, and this situation is a standard one. We recall these three notions (in their both appearances: when the repetitions are allowed and when they are not). We do it by starting with variations without repetitions (validating this decision by a direct relation to cardinality of all finite maps), although one could start with combinations or with permutations. When meeting the permutation, we pay a particular attention to the factorial: we present the approximation formula to calculate it and we extend it (to so-called gamma function).

The number of finite maps, or variations with repetition

Let's recall than

a) any subset of the Cartesian product $A \times V$ is called a relation (in $A \times V$, between elements in A and elements in V); the sentence 'a is in the relation f with v' means that the pair $(a, v) \in f$, and it is commonly written in the form $a\,f\,y$,

b) a function defined in (or, computable on) the set A and assuming values in the set V is the relation f satisfying the condition

$$\{\, y = f(x_1) \;\&\; y = f(x_2) \,\} \Rightarrow x_1 = x_2,$$

where $y = f(x)$ is commonly way to write $x\,f\,y$,

c) a function defined on a m-element set is called a m-element sequence,

d) any n-element sequence formed from elements of m-element set is called a n-element **variation** (**with repetitions**) of this m-element set,

e) the same nomenclature is used to the corresponding map, i.e., a n-element **variation** of a set is the function mapping this set into n-element sequence.

The family of all functions defined on the set A and assuming values in the set V is denoted by V^A [2]. The same symbolism concerns the sequences, so

$$V^A$$

also denotes the collection of all sequences whose elements are in V and are defined on A.

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[1] The term 'combinatorics' was first used in today sense by Gottfried Wilhelm Leibniz in his *Dissertatio de arte combinatoria* (1666).

[2] The denotation V^A was created in a way to call in mind the case taking place with finite mappings (recall that a function from A to V is called a **finite map**, if both A and B are finite). The first example is enough to exhibit a pretty correspondence of the denotation V^A and the cardinalities of finite sets A and V.

Numerical example. Let's deal with 3-element set $A := \{1, 2, 3\}$ and 2-element set $V := \{o, t\}$. Then the family V^A contains $2^3 = 8$ elements, namely

$$W^A = \{ f_1, f_2, \ldots, f_8 \},$$

where functions $f_1, f_2, \ldots, f_8$ are defined as in the Table 1 (e.g., $f_6(2) = o$)

Table 1.

$x =$	1	2	3
$f_1(x) =$	o	o	o
$f_2(x) =$	o	o	t
$f_3(x) =$	o	t	o
$f_4(x) =$	o	t	t
$f_5(x) =$	t	o	o
$f_6(x) =$	t	o	t
$f_7(x) =$	t	t	o
$f_8(x) =$	t	t	t

Historical example. As far as we know, the most ancient problem in combinatorics is that reported, as problem no.79, in Ahmed papyrus written about 1800 BC. It deals with seven houses, $7^2 = 49$ seven cats, $7^3 = 343$ mice and $7^4 = 2401$ ears of spelt (an ear is a kind of the spike, the part of a plant it keeps grain; Latin spicus, French. épi, German Ähre, Polish kłos; a spelt, aka dinkel, *Triticum spelta*, wheat, is an ancient species of wheat, very popular cereal up to mediaval times, now almost not cultivated, in French épeautre, in German: Spelz, Dinkel, in Polish: pszenica orkisz). Author says that if sown, each ear of corn would have produced seven hekat of grain (in ancient Egypt a hekat was a portion of grain, bread, and beer; it equals 4.8 liters in today's measurements), and ask how many hekats are lost. The answer is $7^5 = 16\,807$; in today's language, this is the number of (5, 7)-variations, or 5-element variations of 7-element set.

The similar problem appears in Fibonacci's *Liber abaci* (2012) and in an English nursery rhyme *As I was going to St Ives* (transcribed around 1730, below in version from 1825):

As I was going to St. Ives,
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits:
Kits, cats, sacks, and wives,
How many were there going to St. Ives?

Typical example. 600 € can be distributed between 3 persons or can be sent to nobody. As we can see (saying that, in Table 1, the letters t and o denote that a person obtains a prize or does not, resp.), there are possible 8 situations: no persons takes a prize, a prize is going to one person, the total 600 € is granted to two persons, every person is prized up (with 200 € each, if the division is uniform).

Answering. Obviously, in the considered case instead of the talk about functions we can talk about (finite) sequences, because every function, as definite on finite set, is nothing else than an appropriate sequence: it is 3-element sequence whose elements belong to V, e.g.

$$f_6 = (t, o, t).$$

The observation on the cardinality 2^3 is easily to be extended to arbitrary finite sets A and V, and this way we find that the cardinality of the set V^A is equal

$$|\, V^A \,| = |\, V \,|^{|\, A \,|}.$$

The above can be spoken out as follows: there are w^a sequences of a elements, every one of which is in a w-element set. Using, as it is commonly accepted, letters i, j, k, l, m, n to denote quantities whose values are natural numbers, we say

there exists n^m functions which are defined in m-element set and take values in n-element set,

there exist n^m sequences of m elements which belong to n-element set,

the number of n-element variations (with repetitions) of m-element set is n^m,

there are n^m n-element variations (with repetitions) of m-element set, or

$$\mathrm{card}\{\, (\, v_1, v_2, \ldots, v_n\,) :\; v_1, v_2, \ldots, v_n \in \{\, a_1, a_2, \ldots, a_m \,\}\,\} = n^m.$$

One more typical example. 5 persons mount in the lift standing on the ground floor. This lift takes them up, they get off at least at the highest floor, which is the 8th (here the UK and Polish standard is presented; in USA the ground floor is the same as the 1st floor, so 9-storey building in USA has 9 floors, and in UK and Poland it has 8 floors). How many ways they can get out of the elevator which can stop at every of 8 other floors ?

Answering. Obviously, here we have to deal with one of following maps:

a) to every person there is attached the number of the floor on which this persons gets off the lift,
b) to every floor there is attached the number of persons who get off on this floor.

Case a) is fully described by a sequence

$$(v_1, v_2, v_3, v_4, v_5),$$

where v_j says: 'person no.j gets off on the floor no. v_j'

(and we can memorize it in the phrase: 'a person chooses a floor').

For example, (4, 6, 4, 8, 6) reports that

persons number 1 and 3 get off on 4th floor,
persons no.2 and 5 get off on 6th floor,
person no.4 gets off on the 8th floor.

Since any person can get off on any of 8 floors, so there are

$$8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 8^5 = 32\,768$$

ways 5 persons can get off the elevator which can stop at every of 8 other floors. This is as many as there are 5-element sequences with elements taken from the set {1, 2, …, 8}.

In case b) we have sequences

$$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8),$$

where v_j says: 'on the floor no.j there are v_j persons getting off'

(and we can memorize it in the phrase: 'a floor takes a person').

For example, the sequence

$$(0, 0, 0, 2, 0, 1, 0, 2)$$

says that 1 person gets off on the 3rd floor,
2 persons get off on the 5th floor,
2 persons get off on 8th floor

(so it reports the same arrangement as an exemplary one given above).

Any possible situation is fully described by 8-element sequence $(v_1, v_2, ..., v_8)$, which elements are taken form the set {0, 1, 2, 3, 4, 5} and such that

$$v_1 + v_2 + ... + v_8 = 5.$$

Obviously, there are $5^8 = 390\,625$ sequences $(v_1, v_2, ..., v_8)$ which elements are taken from the set {0, 1 2, 3, 4, 5}. Naturally, the condition imposed on the sum, $v_1 + v_2 + ... + v_8 = 5$, reduces this number, and it is not so easy to know by how much. This makes that we do not apply the description considered here as case b).

Permutations

A **permutation** (**without repetitions**), or a **rearrangement**, or a **substitution**, of a given set is any sequence containing each element of this set exactly once. Thus a permutation of the set $A = \{ a_1, a_2, \ldots, a_n \}$ is the sequence

$$(p_1, p_2, \ldots, p_n)$$

such that $p_j \in A$ for every j and $p_j \neq p_k$ dla $j \neq k$. This sequence is also referred to as n-element permutation.

Beside of the definition given above (called a resulting, or a sequentiual, definition of permutation) one can understood the term ‘permutation’ in its functional meaning: a **permutation** of a set (assumed to be finte) is a bijection of this set onto it. Analogously as in many other cases (e.g., when sine is used to name the concrete function and to name the value of this function for its argument), both meaning of the word ‘permutation’ can be equivalently used,

Permutations were considered in the antiguity , and the number of permutations of n-element set is equal to [3)]

[3)] The fact that a n-element set has n! permutations is noticed in *Lilavati* authored about the year 1150 by Bhaskara II.

Permutations are the main subject of the first chapter (*Caput I. De permutationibus*) of *Ars conjectandi*, the book considered as the first one dedicated to the combinatorics. Its author, Jacob Bernoulli, among various examples deals with 22 monosyllabic words (lex, rex, grex, res, spes, ius, thus, sal, sol, lux, laus, mars, mens, sors, lis, vis, styx, pus, nox, fax, criu, frans). An interesting example of permutations is provided by 8-word Latin hexametr

Tot tibi sunt dotes, Virgo, quot sidera cœlo

(it says: Virgin, your virtues so as many as there are stars in the heaven) composed by Jesuit priest Bernard Bauhuis. In *Pietatis Thaumata* (1617) its author, Erycius Puteanus, a professor at the University of Louvain, listed 1022 permutations, and all of them (e.g. *Tot tibi sunt dotes, Virgo, quot sidera cœlo*; *Sidera quot cœlo, tot sunt Virgo tibi dotes*) tell the glory of Mary, but there is not included the permutation as *Sidera tot cœlo, Virgo, quot sunt tibi dotes* (it can be read as the upper bound on the Virgin’s virtues). Leibniz in *Dissertatio de arte combinatorial* (1666) claimed that Puteanus stopped at 1022, because 1022 was the number of visible stars in Ptolemy’s well-known catalog of the skies.

Much more recent is the factorial symbol, n !, it was introduced in 1808 by Christian Kramp in his *Éléments d'arithmétique universelle*, and disseminated thanks to such works as *Démonstration de la fausseté du théorème énoncé à la page 320 du IX.e volume de ce recueil* (1819) by Pierre-Frédéric Sarrus and the book *System der Mathematik* (1829) by Martin Ohm. Leonard Euler, in *Calcul de la probabilité dans le jeu de recontre* (1751), denoted the product of n successive natural numbers starting at 1 by N, Johann Bernhard Basedow, in *Bewiesene Grundsätze der reinen Mathematik* (1774), put * (e.g., $5^* = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$), in *Mathematische Werke* (1894) Karl Weierstrass put the line above (e.g., $\overline{5} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$). Kramp read n ! as *n Fakultet*, it was adopted in England and Italy as *n factorial* and *n fattoriale*, resp., in Polish language it is read as *n silnia* (from *silny* = strong, powerful).

$$n\,! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

(n ! is read as 'n factorial'), e.g. 8 ! = 40 320.

Let's prove this formula inductively.

a) Base case: Obviously, for $n = 1$ there is $n\,! = 1\,! = 1$ permutation.
b) Inductive step: We assume that n-element set A has n ! permutations and we consider the set $B := A \cup \{ b \}$ which embraces $n + 1$ elements, because it is enriched with $b \notin A$. Given an arbitrary permutation
$$(p_1, p_2, \ldots, p_n)$$
of A, we form a (n+1)-element permutation,
$$(q_1, q_2, \ldots, q_n, q_{n+1})$$
of B, by placing this new element b in front of p_1, between any two neighboring p_k and p_{k+1} ($1 \le k < n$) or after p_n. It showes that the number of permutations of the set B is (n+1) times larger that that of A, i.e., it equals $(n+1) \cdot n$! This closes the inductive step.

Q.E.D.

As we will see later, a permutation of a n-element set is the n-element variation without repetitions of this set.

Typical example. 5 persons can sit down on 5 chairs
in 5 ! = 120 ways if chairs are arranged in a series,
in 4 ! = 24 ways if chairs are arranged in a circle (e.g., round a table).

A permutation $p = (p_1, p_2, \ldots, p_n)$ can be presented in (proposed by L.A.Cauchy) **two-line notation**,

$$p = \begin{pmatrix} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix}.$$

A composition of two permutations of the same set is defined as the permutation of the permutation of this set. The two-line notation helps a lot to find the compositions of permutations. Let's leave it here at one example, where we compose the permutations $p = (2, 7, 6, 5, 4, 3, 1)$ and

$$q = (2, 3, 4, 1, 7, 5, 6).$$

Composing p with q we write

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 5 & 4 & 3 & 1 \end{pmatrix},$$

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 7 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 7 & 6 & 5 & 4 & 3 & 1 \\ 3 & 6 & 5 & 7 & 1 & 4 & 2 \end{pmatrix}$$

and canceling identical rows (the 2nd row in p and the 1st row in q) we see that the composition is the permutation

$$p \circ q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 5 & 7 & 1 & 4 & 2 \end{pmatrix},$$

or, in the standard form (which can be called an one-line notation),

$$p \circ q = (3, 6, 5, 7, 1, 4, 2).$$

Notice that the composition of permutations is not Abelian (i.e., it is not commutative); really, composing in inverse order we have

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 7 & 5 & 6 \end{pmatrix}$$

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 5 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 & 7 & 5 & 6 \\ 7 & 6 & 5 & 2 & 1 & 4 & 3 \end{pmatrix}$$

$$q \circ p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 2 & 1 & 4 & 3 \end{pmatrix},$$

or $\quad q \circ p = (7, 6, 5, 2, 1, 4, 3) \neq p \circ q.$

Cyclic structure of the permutation

Below, in aim to simplify the denotations, we let us to follow on with not very formal way (and it appears by going to analogies and to graphical representations). We know that any permutation of a set X is a bijection from X onto X. Let's denote this permutation by the letter p. If in X there are

- two elements u_1 and u_2 such that $p(u_1) = u_2$ and $p(u_2) = u_1$, then we say that these two elements form a **cycle of length 2**,
- three elements u_1, u_2 and u_3 such that $p(u_1) = u_2$, $p(u_2) = u_3$ and $p(u_3) = u_1$, then we say that these three elements form a **cycle of length 3**.

Analogously we define what does a **cycle of length *m*** (m being a natural number) mean. It includes the case $m = 1$, when we have a **cycle of length 1** (aka a **singleton cycle**): it is formed by an element u such that $p(u) = u$ (and this u is called a **fixed point** of the function p).

It is easy to visually detect cycles in a given n-permutation: we mark elements of given set as numbered points on the plane (numbers run the set $\{1, 2, 3, \ldots, n\}$) and we draw lines directed form the point no. j to the point no. k if $p(j) = k$. It exhibits so-called **cyclic structure** of the permutation p. This structure can be presented by explicit specification of elements forming cycles. For example, in the permutation $p = (2, 7, 6, 5, 4, 3, 1)$ it is easy to find that

- elements 1, 2 and 7 form a cycle of length 3,
- elements 3 and 6 form a cycle of length 2,
- elements 4 and 5 form a cycle of length 2.

This information can be written down by embracing, into the square brackets, elements involved in a cycle (with the order kept, and, usually, started with the element which can be thought of as the smallest one), so

$$p = [1, 2, 7]\,[3, 6]\;\;[4, 5].$$

Identically, $q = (2, 3, 4, 1, 7, 5, 6) = [1, 2, 3, 4]\;\;[5, 7, 6]$.

Obviously, the cyclic structure of the permutation does not depend on the names of elements of the set under permutation. So, abstracting from these names, we can fully describe the cyclic structure by saying how many cycles of particular lengths are. Usually this information is stored via the inscriptions l^m, where l and m stand for the length of cycle and for its multiplicity, resp.; an additional convention is universally accepted: there are no listed l^0. Therefore the permutation p has the cyclic structure (or the **cyclic type**) $3^1 2^2$, and q is of $4^1 3^1$ cyclic type.

The number of ways to arrange n objects in k cycles is denoted by $s_{n,k}$ or (after the proposal given in [Knuth 1989]) by

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

(and verbalized as 'n cycle k').

For example, $n = 4$ objects (here they are numbers 1, 2, 3 and 4) can be permuted into $k = 2$ cycles in $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$ ways, all these 11 permutations of the set {1, 2, 3, 4} are:

[1] [2, 3, 4]; [1] [2, 4, 3];
[2] [1, 3, 4]; [2] [1, 4, 3];
[3] [1, 2, 4]; [3] [1, 4, 2];
[4] [1, 2, 3]; [4] [1, 3, 2];
[1, 2] [3, 4]; [1, 3] [2, 4]; [1, 4] [2, 3].

Before writing down the recurrence formula for $s_{n,k}$ let's look for three particular cases. Obviously, n objects

- form n cycles iff every cycle is a singleton one; this states that $\begin{bmatrix} n \\ n \end{bmatrix} = 1$,
- can form one cycle in so many ways as n person can sit at a round table; we already know that there are $(n–1)!$ ways to seat n people, so $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$.

Given the cycle $[a, b, c]$ the newcomer d
or turns it into one of cycles $[d, a, b, c] = [a, b, c, d]$, $[a, d, b, c]$ or $[a, b, d, c]$,
or does not join it and this way form the permutation $[a, b, c]$ $[d]$;
in other words, the permutation 3^1 is turned into or 4^1 or $3^1 1^1$.

A deeper insight in the consequences caused by the arrival of a new element to $n–1$ elements already forming the permutation lets to produce (see, e.g., [Knuth 1990, p. 247]) the recursion

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}, \; n \in \boldsymbol{N}; \; k = 1, 2, \ldots, n-1,$$

with boundary values

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0}, \; \begin{bmatrix} n \\ n \end{bmatrix} = 1$$

(the last condition was derived above, the first one is formed the recursive formula to be held true for all natural n).

The number $\begin{bmatrix} n \\ k \end{bmatrix}$ is called a (n,k)-**cyclic number**, a **Stirling cycle number**, or a **Stirling number of the first kind**. This nomenclature was proposed by Niels Nielsen's *Handbuch der Theorie der Gammafunktion* published in 1906, and it recalls James Stirling who presented these numbers in his book *Methodus differentialis sive tractatus de summatione et interpolatione serierum infinitarum* (1730).

In the same work Stirling introduced numbers which are subsequently called Stirling numbers of the second kind. A (n, k)-**subset number**, or a **Stirling subset number**, or a **Stirling number of the second kind** is denoted by $\sigma_{n,k}$, by $S_{n,k}$ or (after Karamata's and Salmeri's proposals presented in 1935 and 1962, resp.) by $\left\{ n \atop k \right\}$ (and read: 'n subset k').

$\left\{ n \atop k \right\}$ is the number of ways to partition a n-element set into k nonempty subsets.

For example, the set {1, 2, 3, 4} of $n = 4$ elements can be partitioned in

1 subset in 1 way: {1,2,3,4},

2 subsets

in $\left\{ 4 \atop 2 \right\} = 7$ ways: {1}{2,3,4}; {2}{1,3,4}; {3},{1,2,4}; {4}{1,2,3}; {1, 2}{3,4}; {1, 3}{2,4},

3 subsets

in $\left\{ 4 \atop 3 \right\} = 6$ ways: {1}{2}{3,4}; {1}{3}{2,4}; {1}{4}{2,3}; {2}{3}{1,4}; {2}{4}{1,3}; {3}{4}{1,2},

4 subsets

in $\left\{ 4 \atop 4 \right\} = 1$ ways: {1}{2}{3}{4}.

This completes, , all $\left\{ 4 \atop 1 \right\} + \left\{ 4 \atop 2 \right\} + \left\{ 4 \atop 3 \right\} + \left\{ 4 \atop 4 \right\}$ =15 ways.

Obviously, n objects

- remain unpartitioned (i.e., form the one set) in $\left\{ n \atop 1 \right\} = 1$ way,
- can be partitioned in n subsets in $\left\{ n \atop n \right\} = 1$ way (every object is a singleton).

One can show that there holds true the recursion

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \cdot \left\{ n-1 \atop k \right\} \text{ for } n \in \boldsymbol{N};\ k = 1, 2, \ldots, n-1,$$

with boundary values $\left\{ n \atop 0 \right\} = \delta_{n,0}$, $\left\{ n \atop n \right\} = 1$

(the last condition was derived above, the first one is formed the recursive formula to be held true for all natural n).

Calculating the factorial – Stirling's formula

The factorial grows very much as its argument increases, and the calculation $n\,!$ is tedious for big n. It can be approximate as follows:

$$n\,! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}.$$

for instance, it gives 10! = 3 628 800 ≈ 3 598 695.62, so the relative error of this approximation is (3 628 800 – 3 598 695.62)/ 3 628 800 ≈ 0.0083.

The relative terror decreases as n increases, namely there holds

$$\lim_{n\to\infty} \frac{1}{n\,!}\cdot\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} = 1.$$

This asymptotic equation was produced (on the base of Euler-Maclaurin summation formula and Wallis' formula) in 1754 by James Stirling (he published his result in the paper *Methodus differentialis*, and the discussed approximation is called a **Stirling's formula**). First approach to get this kind of approximation was undertaken 24 years earlier, in 1730, by Abraham de Moivre (1667-1754) and published in *Miscellanea arithmetica*. He started with the observation

$$\ln(n\,!) = \ln 1 + \ln 2 + \ldots + \ln n =$$

$$= \sum_{k=1}^{n} \ln k \approx \int_{1}^{n} \ln x\, dx = (\,x \ln x - x\,)\Big|_{1}^{n} =$$

$$= n \ln n - n + 1 \approx n \ln n - n = n\,(\,\ln n - \ln e\,) = n \ln\frac{n}{e},$$

and it immediately implies that $n\,! \approx \left(\frac{n}{e}\right)^{n}$.

This approximation differs from Stirling's one by the factor $\sqrt{2\pi n} \approx 2.5066\sqrt{n}$. For n = 10 it gives 10 ! ≈ 453 999.30 (so the relative error is c.0.87), for n = 20 it gives $2.16127622075256587\cdot 10^{17}$ instead of 20 ! ≈ $2.43290200817664\cdot 10^{18}$ (so the realtive error is 0.91). With the increase of n the quality of the approximation is getting worst, and it is shown that

$$\lim_{n\to\infty} \frac{1}{n\,!}\cdot\left(\frac{n}{e}\right)^{n} = 0.$$

The Stirling approximation can be specified as follows

$$n\,! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} e^{\beta_n},$$

where $\frac{1}{12\mathrm{n}+1} < \beta_n < \frac{1}{12n}$.

Generalizing the factorial – gamma function

The generalization of factorial is Euler gamma function; after Legendre's proposal it is commonly denoted by the capital Greek letter gamma, Γ, and is defined by the formula

$$\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-t}\, dt$$

where $x > 0$ (and it is also computable for every non-integer negative x).

The integral in this definition is said to be an **Euler integral of the second kind**, it was first considered in 1830 by Leonhard Euler [4].

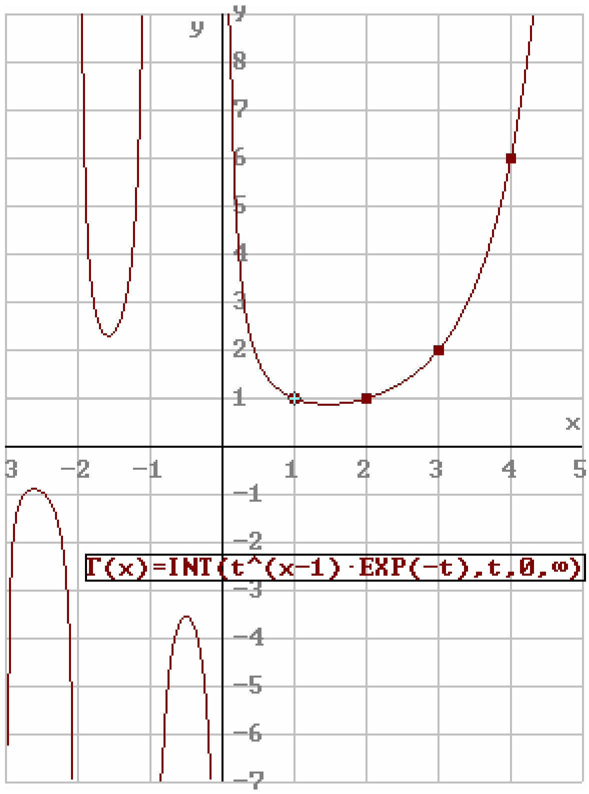

Fig. 1. Graph $y = \Gamma(x)$

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[4] The function $x \to \exp(-x^2)$ is not elementarily integrated, but it can be integrated over the semiaxis Ox. Euler found G when looking for the problem stated in 1720's by Daniel Bernoulli and Christian Goldbach who looked for the extension of the factorial for non-integer argmuents. Originally Euler calculated $\int_0^1 (-\ln y)^n \, dy = 1 \ n\,!$ valid for $n > 0$. Some months before he found that $n\,! = \prod_{k=1}^{n} \left(1+\frac{n}{k}\right)^{-1} \left(1+\frac{1}{k}\right)^{n}$, which is referred to as Euler infinite product and was reported in the paper *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt* sent to St.Petersburg Academy on November 28, 1729.

Applying the integration per partes it is easy to see that

$$\Gamma(x+1) = x \cdot \Gamma(x) \text{ dla } x \notin \{0, -1, -2, -3, \ldots\},$$
$$\Gamma(n) = (n-1)\,! \text{ for any natural } n.$$

The first equality exhibits the relation between the gamma function and the falling power: for $x \in \boldsymbol{N}$ there holds true

$$\Gamma(x) = (x-1)! = 1^{\underline{x-1}}.$$

The last relation tells that the function Γ is a generalization of the factorial. For $x = 1/2$ it gives

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t}\, dt \overset{t=s^2}{=} 2 \cdot \int_0^{+\infty} e^{-s^2}\, ds = \sqrt{\pi}\,.$$

The integral

$$G := \int_0^{+\infty} e^{-x^2}\, dx$$

is known as an Euler-Poisson integral, or a Gauss integral.

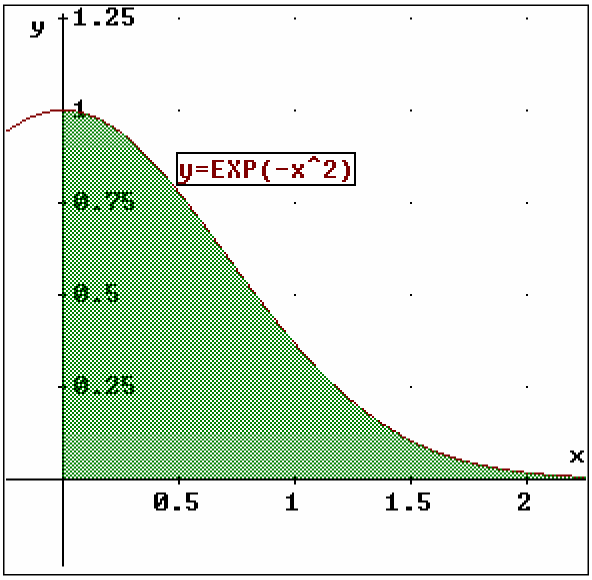

Fig.1. Graph $y = \exp(-x^2)$ and the region which area is G

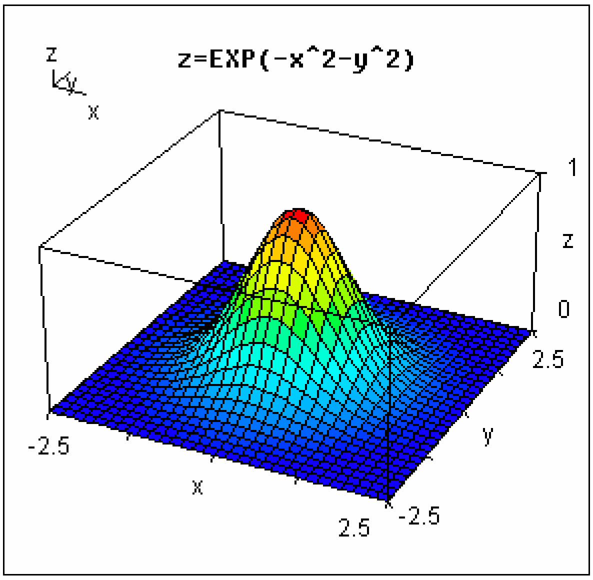

Fig.2. Surface $z = e^{-(x^2+y^2)}$ spread over the rectangle <–2.5, 2.5>×<–2.5, 2.5>

The value G can be obtained via the calculation of the product $G^2 = G \cdot G$ in the Cartesian plane Oxy and switching for the polar coordinate system $Or\theta$. Really, taking into account the quadrant

$$Q_R := \{\, (x, y): 0 \le x;\ 0 \le y;\ x^2 + y^2 < R^2 \,\} = \{\, (r, \theta) : 0 \le r < R;\ 0 \le \theta \le \pi/2 \,\}$$

and sending R to $+\infty$ we see that

$$G^2 = G \cdot G = \int_0^{+\infty} e^{-x^2}\, dx \,.\, \int_0^{+\infty} e^{-y^2}\, dy = \int_0^{+\infty} \left(\int_0^{+\infty} e^{-x^2} dx \right) e^{-y^2}\, dy =$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dx\, dy = \lim_{R\to\infty} \int_0^{\frac{\pi}{2}} \int_0^{R} e^{-r^2} r\, dr\, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} d\theta \cdot \lim_{R\to\infty} \int_0^{R} e^{-r^2} r\, dr = \theta \Big|_0^{\frac{\pi}{2}} \cdot \lim_{R\to\infty} \frac{-e^{-r^2}}{2} \Bigg|_0^{R} =$$

$$= \frac{\pi}{2} \cdot \lim_{R\to\infty} \frac{1-e^{-R^2}}{2} = \frac{\pi}{4},$$

hence $G = \int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

Euler ascent numbers

Given a permutation $(p_1, p_2, \ldots, p_n)$, the index j such that $p_j < p_{j+1}$ is called an **ascent**, and such that $p_j > p_{j+1}$ is called a **descent**.

For instance, the permutation $p = (2,0,1,4,5)$ has 3 ascents (positions no. 2, 3 and 4) and 1 descent (position no.1), and it is easily seen when one draws a polygonal line having vertices (i, p_i), here they are (1,2), (2,0), (3,1), (4,4), (5,5).

The number of permutations of n-element set with k ascents (as well as the number of permutations with k descents) is equal to $A_{n,k}$, where

$$A_{n,0} = A_{n,n} = 1 \text{ for } n = 0, 1, 2, \ldots,$$

$$A_{n,k} = (n-k) \cdot A_{n-1,k-1} + (k+1) \cdot A_{n-1,k} \text{ for } k = 1, 2, \ldots, n-1.$$

These relations were presented by Leonhard Euler in the book *Institutiones calculi differentialis* (1755). The number $A_{n,k}$ is also denoted as

$$\left\langle {n \atop k} \right\rangle,$$

and it is called a (n, k)-th **Euler number** (of the first kind), or **Euler ascent number**, or **Euler ascential number**.
Directly from the combinatorial definition of $A_{n,k}$ it follows that

$$\sum_{k=0}^{n} A_{n,k} = n!,$$

There are showed

a) the symmetry: $A_{n,n\text{-}1\text{-}k} = A_{n,\,k}$,

b) the closed form: $A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$,

c) the relation to Bernoulli numbers:

$$\sum_{k=0}^{n-1} (-1)^k A_{n,k} = \frac{2^{n+1}(2^{n+1}-1)}{n+1} B_{n+1} \text{ for } n \ge 1.$$

Similarly as binomial coefficients can be arranged in Pascal triangle, Euler numbers can be arranged in so-called **Euler ascential triangle**. First seven rows of this triangle are (in line no. n there are listed successive $A_{n,0}$, $A_{n,1}$, $A_{n,2}$, … $A_{n,n}$): $n = 0$: 1;

$n = 1$: 1, 1;
$n = 2$: 1, 4, 1;
$n = 3$: 1, 11, 11, 1;
$n = 4$: 1, 26, 66, 26, 1;
$n = 5$: 1, 57, 302, 302, 57, 1;
$n = 6$: 1, 120, 1191, 2416, 1191, 120, 1;
……………………………

Euler ascent numbers appear in the probability theory when we deal with independent random variables $X_1, X_2, \ldots, X_n$, every one of which is uniformly distributed on <0, 1> and we ask for the probability that the sum $X_1 + X_2 + \ldots + X_n$, assumes its value in the interval <k, k+1>. Then (see, e.g., 1982 Hensley - *Eulerian numbers and the unit cube*, The Fibonacci Quarterly)

$$\Pr\left\{ \sum_{j=1}^{n} X_j \in < k, k+1 > \right\} = \frac{1}{n!} \cdot \left\langle {n \atop k} \right\rangle .$$

Permutation with repetitions

Any sequence of n elements belonging to a set its called a **permutation with repetitions** of this set. Thus a permutation with repetitions allowed of the set $A = \{ a_1, a_2, \ldots, a_n \}$ is a sequence

$$(p_1, p_2, \ldots, p_n)$$

where $p_j \in A$ for every j.

Immediately form the formula for the number of permutations of the set $\{ 1, 2, 3, \ldots, n \}$ it follows that there are

$$\frac{n!}{\beta_1!\beta_2!\ldots\beta_n!}$$

permutations in which the number j repeats β_j times; obviously, $\beta_1 + \beta_2 + \ldots + \beta_n = n$. The discussed expression is sometimes referred to as a multinomial coefficent.

Typical example. The word MATEMATYKA (in Polish: mathematics) is composed of 6 different letters, M, A, T, E, Y, K, the first three of them are repeated 3, 2 and 2 times, resp. This word can be seen as a 10-element sequence (M, A, T, E, M, A, T, Y, K, A). The collection these 10 letters can form

$$\frac{n!}{\beta_1!\beta_2!\ldots\beta_n!} = \frac{10!}{3!\cdot 2!\cdot 2!\cdot 1!\cdot 1!\cdot 1!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 2\cdot 2} = 151\,200$$

sequences (and between them there is only having a meaning).

Variation without repetition

We already discussed variations with (m, n)-repetitions, i.e., n-element sequences whose elements are taken form a m-element set (and we know there are n^m such sequences). When $m > n$ and repetitions are not allowed, we have sequences called variations without repetitions. So, given natural numbers m and n, $m > n$, a n-element **variation without repetitions** of m-element set is any sequence of n elements of m-element set.
In symbols, a n-element variation without the repetitions of the set

$$A = \{ a_1, a_2, \ldots, a_m \}$$

is any sequence

$$(v_1, v_2, \ldots, v_n)$$

such that $v_1, v_2, \ldots, v_n \in A$ and $v_j \neq v_k$ for $j \neq k$.

It is show that for m and n, $m > n$, there exist

$$(m)_n := m \cdot (m-1) \cdot \ldots \cdot (m-n+1) = \frac{m!}{n!}$$

n-element variations without repetitions of m-element set.

In purely mathematical description:

$$\text{card} \{ (v_1,v_2,\ldots,v_n):\ v_1,v_2,\ldots,v_n \in \{ a_1,a_2,\ldots,a_m \} \ \&\ v_j \neq v_k \text{ for } j \neq k \} = \frac{m!}{n!}.$$

Typical example. There are 7 people considered to take 3 places in the board of an enterprise, which are president, vice-president and auxiliary vice-president (aka second vice-president) positions. The election procedure states that there is a single voting and these positions are taken accordingly to the support measured by the numbers of votes (the highest number ensures the president position, the second largest numbers gives the vice-presidency, the third largest numbers results with the second vice-president position). How many personal configurations of the board are possible ?
Answering. There are picked up 3 persons out of 7, and the order is important (you can think that first the president is chosen, next the vice-president, and, at last, the 2nd vice-president). It undergoes the variation, and it is the variation without repetitions. Therefore the answer is $\frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 840$.

This answer can be obtained without evoking the formula for variations without repetition, and we will mention it when talking about combinations.

Combination

A k-element combination, or a k-**combination**, of the n-element set is every k-element subset of this set. So, a k-element combination of the set $A = \{ a_1, a_2, a_3, \ldots, a_n \}$ is any subset

$$\{ b_1, b_2, \ldots, b_k \},$$

where $b_j \in A$.

Every k-element combination of n-element set is the result of the selection of k elements out of n elements. Equivalently, this is one of possible ways at which k objects are taken out of n objects.

Applying mathematical induction one can prove that

$$\text{card} \{ B : B \subset \{ a_1, a_2, a_3, \ldots, a_n \} \} = \binom{n}{k},$$

where

$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1} = \frac{n^{\underline{k}}}{k!}.$$

This quantity is called a **combinatorial symbol** and is read as 'n choose k' (although one can still say 'n off k' or 'n over k'). The notation with parenthesis was introduced, in 1826 in the textbook *Die combinatorische Analyse*, by Andreas von Ettingshausen. In India the problems involving the number of combinations was discussed in the 6th century: in the chapter on perfumes in Varahamihira's *Brhatsamhita* there is written that by taking any four of sixteen given ingredients and mixing them in various proportions there can be produced 96 different scents; unfortunately, no calculations are presented and there are listed no requirements on the final products, we can only believe than no all 1820 possibilities of mixing, in identical proportions, are accepted. More details provide Brahmagupta (c.628) and Bhaskara in *Lilivati* (c.1150) where there are taken into account, a.o., six different tastes (sweet, pungent, astringent, sour, salt and bitter) to obtain a mix of some of them. As far as we know, the first expression of the formula in almost modern notation and in full generality occurs in *Cursus mathematicus* (all five volumes were published in 1634-37) of Herigonus (Pierre Hérigone).

Typical example. Given the set $\{a, b, c, d, e\}$, we can chose its 2-element subsets in $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$ ways. In other words: there are ten 2-element subset of the set $\{a, b, c, d, e\}$, there are ten 2-element combinations of letters a, b, c, d and e. All these subsets are: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$,

$\{b, c\}$, $\{b, d\}$, $\{b, e\}$,

$\{c, d\}$, $\{c, e\}$,

$\{d, e\}$.

It is not difficult to state that there are $\binom{k+z}{k}$ sequences in which the value 1 occurs k times, and the value 0 occurs z times.

As it was already mentioned, Blaise Pascal noticed that numbers forming his arithmetic triangle (now known as Pascal's one) can be calculated after the above formula, in consequence they are combinatorial coefficients. This way we see combinatorial coefficients and binomial coefficients are the same objects,

$$C_{n,k} = \binom{n}{k}.$$

This can be proved by inspecting so-called boundary values of both quantities (they are that for $k = 0$ and $k = n$, in both cases equal to 1) and by the transforming the left side of the Pingala recurrence, consecutively,

$$\begin{aligned}
C_{n-1,k-1} + C_{n-1,k} &= \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)\,!}{(k-1)\,!\cdot(n-k)\,!} + \frac{(n-1)\,!}{k\,!\cdot(n-1-k)\,!} = \\
&\frac{(n-1)\,!}{(k-1)\,!\cdot(n-k)\cdot(n-k-1)\,!} + \frac{(n-1)\,!}{k\cdot(k-1)\,!\cdot(n-1-k)\,!} = \\
&\frac{(n-1)!}{(k-1)\,!\cdot(n-k-1)\,!}\cdot\left\{\frac{1}{n-k} + \frac{1}{k}\right\} = \frac{(n-1)!}{(k-1)\,!\cdot(n-k-1)\,!}\cdot\frac{n}{(n-k)\cdot k} = \\
&\frac{n\,!}{k\ \,!\cdot(n-k)\,!}\cdot\frac{n}{(n-k)\cdot k} = \binom{n}{k} = C_{n,k}
\end{aligned}$$

Q.E.D.

Multiset numbers

Recall that $\{a, a\} = \{a\}$ for any a. This illustrates a general approach in the set theory that every set with repetitive elements is equal to the set understood (in its classical sense, i.e.) as a collection of different elements. A generalization of (classical) sets are sets where all identical elements are taken into it account not once, but they are taken into account as many time as they are (so the repetitions are allowed). For example,

a) given two different elements, a and b, they can form four sets of the cardinality 3, namely $\{a, a, a\}$, $\{a, a, b\}$, $\{a, b, b\}$, $\{b, b, b\}$,
b) given a set $\{a, b, c\}$, we can form six 2-element sets with elements being allowed to be repeated; these sets are:
$$\{a, a\},\ \{a, b\},\ \{a, c\},\ \{b, b\},\ \{b, c\},\ \{c, c\}$$

Hoping that this example is illustrative enough, we do not give here a formal definition of such sets, we only say that any one of them (of cardinality n and composed by elements taken from a k-element set) is called a **multiset**, or a **bag**.

Commonly a multiset composed of m different elements, in which element no. j repeats r_j times (so j runs from 1 through m) is denoted by

$$\left\{ 1^{r_1}, 2^{r_2}, \dots, j^{r_j} \right\}.$$

The number of multisets of cardinality n and generated by k elements is denoted by $\left(\!\!\binom{n}{k}\!\!\right)$, it is called a **multiset number**, or a **multiset coefficient**, and is read as 'n multichoose k' (and it is analogous to the way the binomial coefficient is read 'n choose k').

It is not difficult to state that there are

$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k} = \frac{n\cdot(n+1)\cdot\ldots\cdot(n+k-1)}{k\cdot(k-1)\cdot\ldots\cdot 1} = \frac{n^{\overline{k}}}{k!}.$$

In other words: the multiset number is the number of ways to choose k elements from a n-element set if repetitions are allowed.

Pingala recursion for multiset numbers reads as follows:

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k}\!\!\right) \text{ for } n, k \in \boldsymbol{N},$$

$$\left(\!\!\binom{n}{0}\!\!\right) = 1 \text{ for } n \in \boldsymbol{N}, \quad \left(\!\!\binom{0}{k}\!\!\right) = 0 \text{ for } k \in \boldsymbol{N}.$$

One can calculate $\left(\!\binom{n}{k}\!\right) = (-1)^k \cdot \binom{-n}{k}$,

so multiset numbers are coefficients in the expansion

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \cdot (-x)^k = \sum_{k=0}^{\infty} (-1)^k \cdot \binom{-n}{k} \cdot x^k = \sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) \cdot x^k .$$

DRAFT:

Example (reporting so-called first bars and stars problem). There are 7 undistinguished coins to be given to 3 persons (say: Ann, Bill and Curtis).
We can represent any coin as a star, *, so we have seven stars * * * * * * *. A way these coins go to these three persons can be shown by inserting 2 bars into the star sequences. Examples of this action are

* * * | * * | * * (Ann gets 3 coins, Bill gets 2 coins, Curtis gets 2 coins),
* | * * * * * * | (Ann gets 1 coin, Bill gets 6 coins, Curtis get no coin).

The distribution of coins among persons is represented by inserting $k = 2$ bars into the sequence of $n = 7$ stars.
There are $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{7}{2}\!\right) = \binom{7+2-1}{2} = \binom{8}{2} = 28$ ways to distribute coins.

Example (reporting so-called second bars and stars problem). Seven indistinguishable coins are to be distributed among Ann, Bill and Curtis so that each of them receives at least one coin. We can represent any coin as a star, *, so we have seven stars * * * * * * *. A way these coins go to these three persons can be shown by inserting 2 bars into the star sequences. Examples of this action are

* * * | * * | * * (Ann gets 3 coins, Bill gets 2 coins, Curtis gets 2 coins),
* | * * | * * * * (Ann gets 1 coin, Bill gets 2 coins, Curtis gets 4 coins).

There are $\binom{7-1}{3-1} = \binom{6}{2} = 15$ ways to distribute coins.

Four families of sequences

Stirling cycle numbers, Stirling subset numbers and Euler ascent numbers decompose $n!$, i.e., $\sum_{k=0}^{n}\left\{{n \atop k}\right\} = \sum_{k=0}^{n}\left[{n \atop k}\right] = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle = n!$.

In consequence, when divided by $n!$, these numbers decompose 1. So,

$$\sum_{k=0}^{n}\binom{n}{k} = \sum_{k=0}^{n}\frac{1}{n!}\cdot\left\{{n \atop k}\right\} = \sum_{k=0}^{n}\frac{1}{n!}\cdot\left[{n \atop k}\right] = \sum_{k=0}^{n}\frac{1}{n!}\cdot\left\langle{n \atop k}\right\rangle = 1.$$

We will deal with these sequences when talking about discrete finite distributions (and we will say that these sequences are probability mass functions of specific random variables).

Now let's notice another common property of considered four families: numbers forming any one of these sequences satisfy similar recurrences, namely

$$\text{combinatorial coefficients: } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

$$\text{Stirling cycle numbers: } \left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\cdot\left[{n-1 \atop k}\right],$$

$$\text{Stirling subset numbers: } \left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\cdot\left\{{n-1 \atop k}\right\},$$

$$\text{Euler ascent numbers: } \left\langle{n \atop k}\right\rangle = (n-k)\cdot\left\langle{n-1 \atop k-1}\right\rangle + (k+1)\cdot\left\langle{n-1 \atop k}\right\rangle$$

for $n \in \{1, 2, 3, 4, 5, 6, 7, \ldots\}$ and $k = 1, 2, 3, \ldots, n–1$,

with boundary values $\binom{n}{0} = \binom{n}{n} = 1,$

$$\left[{n \atop 0}\right] = \left\{{n \atop 0}\right\} = \left\langle{n \atop 0}\right\rangle = \delta_{n,0},$$

$$\left[{n \atop n}\right] = \left\{{n \atop n}\right\} = \left\langle{n \atop n}\right\rangle = 1,$$

where $n = 0$ is admitted.

The first recurrence is simply the Pingala recurrence (for binomial coefficients). Other recursive formulas are also called **Pingala recurrences** (for Stirling numbers and for Euler ascent numbers, resp.). We already noticed that first three recurrences are associated to coefficients of the representations of appropriate powers in appropriate bases (recall the name of these bases: a Stevin basis, a falling power basis, a rising power basis). An analogous representation exists also for Euler ascent numbers.

$\binom{n+k-1}{k}$ sposobów, na które spośród n elementów można wybrać k elementów niekoniecznie różnych; każdy z tych sposobów nazywamy k-elementową **kombinacją z powtórzeniami** zbioru n elementowego,

$\binom{k+z}{k}$ ciągów zero-jedynkowych, w których jest k jedynek i z zer,

$\binom{z+1}{k}$ ciągów, których wyrazami jest k jedynek, z zer i gdzie sąsiadem jedynki nie jest jedynka, sąsiadem zera nie jest 0: http://en.wikipedia.org/wiki/Binomial_coefficient ????

Przykłady.

Przykład 2. Dla zbioru $\{1, 2, 3, 4\}$ kombinacji 2-elementowych z powtórzeniami jest

$$\binom{4+1-2}{2} = \binom{3}{2}\frac{5!}{2!\cdot 3!} = \frac{5\cdot 4}{2\cdot 1} = 10,$$

są to zbiory: $\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\},$
$\{2, 2\}, \{2, 3\}, \{2, 4\},$
$\{3, 3\}, \{3, 4\},$
$\{4, 4\}.$

Przykład 3. Ciągów, których jedynymi wyrazami są 3 jedynki i 4 zera, jest

$$\binom{n+k}{k} = \binom{4+3}{3} = \binom{7}{3} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35.$$

Przykład 4. Ciągów, których jedynymi wyrazami są 3 jedynki i 4 zera, jest

$$\binom{z+1}{k} = \binom{4+1}{3} = \binom{5}{3} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4}{2\cdot 1} = 10$$

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Inaczej

Jeżeli zbiory A i W są skończone, to

ak wia

.

Kombinacje (combinations)

k-elementową **kombinacją bez powtórzeń** (ze) zbioru n-elementowego ($k \leq n$) nazywamy każdy podzbiór k-elementowy tego zbioru.
Kombinacji bez powtórzeń k-elementowych ze zbioru n-elementowego jest

$$\binom{n}{k} = \frac{n\,!}{k\,! \cdot (n-k)\,!};$$

Ponieważ ta liczba występuje we wzorze dwumiennym Newtona, nazywa się ją współczynnikiem dwumiennym , a także współczynnikiem binominalnym. Często oznacza się ją przez comb(n, k).

Przykład. Dla zbioru $\{a, b, c, d, e\}$ kombinacji 2-elementowych jest

$\binom{5}{2} = \frac{5\,!}{2\,! \cdot 3\,!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$, są to zbiory:

$$\begin{aligned} &\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \\ &\qquad \{b, c\}, \{b, d\}, \{b, e\}, \\ &\qquad\qquad \{c, d\}, \{c, e\}, \\ &\qquad\qquad\qquad \{d, e\}. \end{aligned}$$

k-elementową **kombinacją z (ewentualnymi) powtórzeniami** (ze) zbioru n-elementowego ($k \leq n$) nazywamy każdy zestaw k elementów, z których każdy należy do tego zbioru.

Takich kombinacji jest $\binom{n+1-k}{k} = \frac{(n+1-k)\,!}{k\,! \cdot (n-1)\,!}$

Przykład. Dla zbioru $\{1, 2, 3, 4\}$ kombinacji 2-elementowych z powtórzeniami jest

$$\binom{4+1-2}{2}=\binom{3}{2}\frac{5\,!}{2\,!\cdot 3\,!}=\frac{5\cdot 4}{2\cdot 1}=10$$ 10, są to zbiory:

{1, 1}, {1, 2}, {1, 3}, {1, 4},
{2, 2}, {2, 3}, {2, 4},
{3, 3}, {3, 4},
{4, 4}.

Permutacje (permutations)

Permutacją (**bez powtórzeń**) zbioru n-elementowego nazywamy każdy ciąg n-wyrazowy utworzony ze wszystkich elementów tego zbioru.
Liczba permutacji zbioru n-elementowego jest równa $n!$.

Permutacją z powtórzeniami zbioru n-elementowego nazywamy każdy ciąg n-wyrazowy utworzony z elementów tego zbioru, wśród których pewne elementy powtarzają się odpowiednio $n_1, n_2, \ldots, n_s$ razy.
Liczba permutacji z powtórzeniami zbioru n-elementowego, wśród których pewne elementy powtarzają się odpowiednio $n_1, n_2, \ldots, n_s$ razy, jest równa

$$\frac{n\,!}{n_1!\cdot n_2!\cdot\ldots\cdot n_s!}\,.$$

Wariacje (variations with and without repetitions)

k-elementową **wariacją** (**bez powtórzeń**) zbioru n-elementowego ($k \le n$) nazywamy każdy k-wyrazowy ciąg różnych elementów tego zbioru.
k-elementowych wariacji bez powtórzeń zbioru n-elementowego jest

$$\frac{n\,!}{(n-k)\,!}\,.$$

k-elementową w**ariacją z powtórzeniami** zbioru n-elementowego ($k \le n$) nazywamy każdy k-wyrazowy ciąg różnych lub nieróżniących się elementów tego zbioru. Takich wariacji jest

$$n^k\,.$$

http://www.cs.xu.edu/math/Sources/Combinations/index.html

Combinations

The computations of probabilities often depend upon the proper enumeration of outcomes. As such it requires the theory of permutations and combinations as the foundation. This section outlines the development of the theory. Of course, given n distinct objects, the number of permutations is equal to n factorial, i.e.,

$$n! = n\cdot(n–1)\cdot(n–2)\cdot\ldots\cdot2\cdot1;$$

this result was . The number of ways to select k objects from among these n, that is, the number of combinations of size k, is $C(n,k) = n!/[k!(n-k)!]$. Two important variations to these problems are to permit the permutations to have a shorter length than n and to relax the requirement that the objects appear to be distinct. Less obvious is the connection of these counting problems to the number of divisors of a number.

O permutacjach pisał J Bernoulli w Ars conjectandi, poświęcając im rozdział pierwszy (Caput I. De permutationibus). Wśród przykładów, jakie rozważa, znajdujemy ten, gdzie mamy do czynienia z 22 wyrazami jednosylabowymi

lex, rex, grex, res, spes, ius, thus, sal, sol, lux, laus oraz mars, mens, sors, lis, vis, styx, pus, nox, fax, criu, fraus,

Following The art of computer programming: Generating all trees (Knuth …, 7.2.1.7;), Let's take 8-word Latin hexametr

Tot tibi sunt dotes, Virgo, quot sidera cælo

It says: Virgin, your virtues so as mamy as there are stars in the heaven, it was composed by Jesuit priest Bernard Bauhuis

its 1022 permutations presented, in Pietatis Thaumata (1617) Erycius Puteanus, a professor at the University of Louvain; examples of these permutations are:

Tot tibi sunt dotes, Virgo, quot sidera cælo

Sidera quot cælo, tot sunt Virgo tibi dotes,

There is not included the permutation as the following omne

Sidera tot cælo, Virgo, quot sunt tibi dotes,

since it can be read as the upper bound on the Virgin's virtues.

8 ! = 40 320

Leibniz in Dissertatio de arte combinatorial (1666)

Puteanus stopped at 1022, because 1022 was the number of visible stars in Ptolemy's well-known catalog of the skies.

PERMUTATION. Leibniz used the term *variationes* and Wallis adopted *alternationes* (Smith vol. 2, page 528).

In 1678 Thomas Strode, *A Short Treatise of the Combinations, Elections, Permutations & Composition of Quantities,* has: "By Variations, permutation or changes of the Places of Quantities, I mean, how many several ways any given Number of Quantities may be changed." [OED]

Lexicon Technicum, or an universal English dictionary of arts and sciences (1710) has: "Variation, or Permutation of Quantities, is the changing any number of given Quantities, with respect to their Places." [OED]

According to Smith vol. 2, page 528, *permutation* first appears in print with its present meaning in *Ars Conjectandi* by Jacques Bernoulli: "*De Permutationibus. Permutationes* rerum voco variationes..." This seems to be incorrect.

John Wallis

Todhunter first cites the English mathematician John Wallis (1616-1703) who's de Algebra Tractatus was published in 1693. This work is prefaced by a history of the subject. Of interest here is the material quoted from William Buckley. Wallis says:

"Libet, hac occasione, dum de Combinationibus agitur; hic subjungere *Regulam Combinationis* quam habet *Guilielmus Buclaeus*, Anglus, in Arithmetica sua, versibus scripta, ante annos plus minus 190; quae ad calcem Logicae *Joahnnis Seatoni* subjicitur, in Editione *Cantabrigiensi*, ante annos quasi 60. (sed medose:) Consonam Doctrinae de Combinationibus supra traditae, quam ego publicis Praelectionibus exposui *Oxoniae*, Annis 1671, 1672." (page 489)

This work by Buckley is appended to the Dialectica of John Seton published in 1584. The verses explaining arithmetic are on pages 261-275.

In *The Doctrine of Combinations and Permutations* published by Francis Maseres in 1795 may be found an English translation of the part of the Algebra entitled "Of Combinations, Alternations and Aliquot Parts" (pages 271-351). We quote the corresponding paragraph to the passage given above:

"I shall subjoin to this Chapter (as properly appertaining to this place,) an Explication of the *Rule of Combination*, which I find in *Buckley*'s Arithmetick, at the end of *Seaton*'s Logick, (in the Cambridge edition;) which (because obscure,) Mr. George Fairfax (a Teacher of the Mathematicks then in Oxford,) desired me to explain; to whom (Sept. 12, 1674,) I gave the explication under written; Consonant to the doctrine of this Treatise, (which had been long before written, and was the subject of divers public Lectures in Oxford, in the years 1671, 1672.)"

The chapters of Wallis are

1. Of the variety of Elections, or Choice, in taking or leaving One or more, out of a certain Number of things proposed.

2. Of Alternations, or the different change of Order, in any Number of things proposed.
3. Of the Divisors and Aliquot parts, of a Number proposed.
4. Monsieur Fermat's Problems concerning Divisors and Aliquot Parts.

The Tot Tibi Variations

Apparently, some gentlemen of the 17th century studied permutations as a diversion. By Erycius Puteanus (in Flemish, Vander Putten and in French, Henry Dupuy (1574-1646)) we have the book *Eryci Puteani Peietatis Thaumata in Bernardi Bauhusii è Societate Jesu Proteum Parthenium*. Bernardus Balhusius of Belgium (Bernard Balhuis, 1575-1619), to whom this refers, was a Jesuit priest who is most well-known for his 5 books of epigrams. The most famous of these is the one composed in honor of the Virgin Mary.

Tot tibi sunt dotes, Virgo, quot sidera caelo.

As many qualities are yours, Virgin, as stars in the sky.

Puteanus lists a total of 1022 permutations of the words. This number was chosen to correspond to the number of stars in Ptolemey's catalog. While still preserving meter, Wallis increased the number of permutations to 3096 and Jakob Bernoulli even higher to 3312.

Another line

Rex, Dux, Sol, Lex, Lux, Fons, Spes, Pax, Mons, Petra, Christus

according to Balhuis admits 3,628,800 permutations while preserving meter. Wallis corrected this to 3,265,920.

Blaise Pascal

Advancement in the study of combinations occurred with the treatise on The Arithmetic Triangle by Blaise Pascal. This work was made available in 1665, but had been printed in 1654.

Franz van Schooten

On pages 373-403 of the Exercitationes Mathematicorum Schooten introduces the section "Ratio inveniendi electiones omnes, quae fieri possunt, data multitudine rerum." That is, the reckoning of discovering all choices, which are able to happen, given from many things.

Gottlieb Leibniz

In 1666 was published the Dissertatio de ArteCombinatoria of Leibniz. This is the earliest work of him connected to mathematics, but it is of little interest. More properly it should be classified among his philosophical works. We do note that Leibniz does include in Problem I: "Dato numero et exponente complexiones invenire" the arithmetic triangle and uses it to compute the number of combinations of various sizes. In Problem VI: "Dato numero rerum, variationes ordinis invenire" he computes the number of permutations of 24 objects and also examines the permutation of phrases. For example, he quotes the lines from Thomas Lansius of Tübingen (Thomas Lanß 1577-1657):

Lex, Rex, Grex, Res, Spes, Jus, Thus, Sol, Sol (bona), Lux, Laus

Mars, Mors, Sors, Lis, Vis, Styx, Pus, Nox, Fex (mala), Crux, Fraus

which contain 11 monosyllables and thus admit 11! = 39,916,800 permutations.

Lesser Figures

Jean Prestet (1648-1691) has given in 1689 *Nouveaux éléments de mathématiques* in Volume I and Volume II. In Volume I, Book V treats of Combinations and Permutations.

André Tacquet (1612-1660), a Flemish Jesuit, has given Arithmeticae Theoria et Praxis, published at Bruxellis in 1655. The edition linked is the corrected version dated 1683. Pages 49-51 treat briefly of permutations. In Book V, Chapter 8, pages 375 - 383 concern permutations and combinations.

Pierwszym, który rozważał permutacje w kontekście szerszym niż jedynie przestawienia, był Joseph-Louis Lagrange – pracując nad kwestią rozwiązalności równania algebraicznego stopnia 5 (i nawiązując do uwag, jakie poczynił Évariste Galois), stworzył teorię grup permutacji (*Réflexions sur la théorie algébrique des equations*, 1770-71). Wyniki te rozszerzył Paolo Ruffini (*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*, 1799), w szczególności przedstawiając dowód na nierozwiązalność tego równania w skończonej liczbie kroków; dowód Ruffiniego zawierał pewne luki, uzupełnił go Niels Henryk Abel w 1824 i odtąd mówi się o twierdzeniu Abla-Ruffiniego.